A laminar planetary jet

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A theory of a jet in a rotating, viscous fluid is developed. It is suggested that it may be related to a jet observed in an experiment with a rotating spherical shell of liquid and, in addition, may partly explain the existence of the subsurface equatorial current in the Pacific Ocean. The fundamental physical idea is that of a balance of vorticity brought into the jet by advection and diffused by friction. The necessary approximations are borrowed from boundary-layer theory. The linear case is solved completely and inferences about the non-linear jet are obtained by dimensional reasoning.

1. Introduction

One of the intriguing features of large-scale flow patterns in the oceans is the tendency for some of the surface and subsurface currents to concentrate in narrow strips or jets. The best known of these is the Gulf Stream, but a few observations have been made recently of a westerly current of planetary scale in the Pacific several hundred feet below the surface at the Equator. It is called the equatorial undercurrent or Cromwell current and is described in a number of papers (Cromwell, Montgomery & Stroup (1954), Knauss & King (1958), Hidaka & Nagata (1958)). Briefly, it is an eastward moving current of water 300 km wide symmetrically located with respect to the equator. The maximum speed is about 3 knots. Its longitudinal extent is still in doubt but it extends at least 3000 miles west of the Galapagos Islands.

Velocity concentrations occur also in the atmosphere and in certain laboratory experiments. One of interest to this paper was found in an experiment by the author (Long, 1952) with a shell of water between two rigidly connected concentric hemispheres in rotation. An obstacle is moved along a latitude circle in the water in the direction of the rotation but either slower or faster. If the obstacle rotates slower than the fluid, a series of waves develops around the globe as in figure 1. If the obstacle moves faster than the fluid, it drags around with it the westerly jet† of figure 2.

A plausible explanation for the jet in figure 2 (and a parallel explanation for the absence of the jet in figure 1) may be developed from the schematic drawing in figure 3. We suppose that the motion in the shell is two-dimensional, i.e. parallel to spherical surfaces and with no variations in directions normal to these surfaces. Ahead of the obstacle the fluid moves more slowly than the barrier and speeds up

† This is really wake flow rather than jet flow, but the distinction is not important for our present purposes.

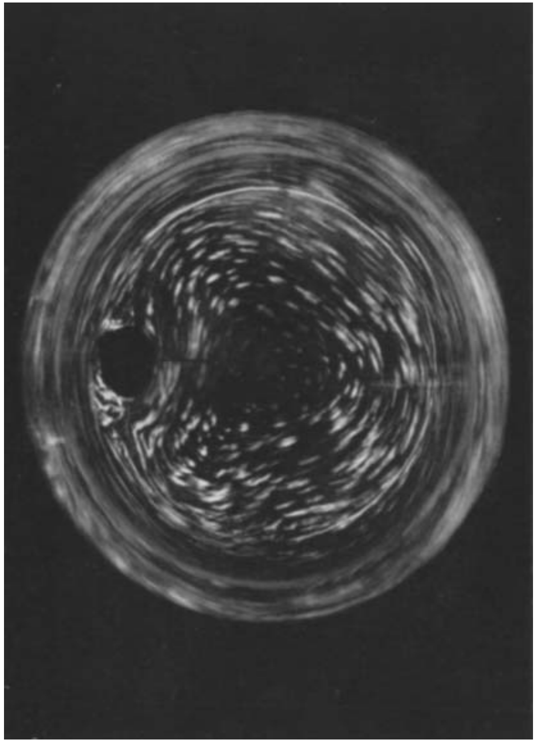

FIGURE 1. Waves in the westerlies. In this experiment a spherical shell of water in basic solid rotation Ω moves around an obstacle rotating at a fixed latitude at an angular rotation $\omega < \Omega$. The motion is with respect to the obstacle. In meteorological language, if the water represents the atmosphere, the effect is that of a westerly wind blowing past a barrier extending through the depth of the atmosphere at 45° lat.

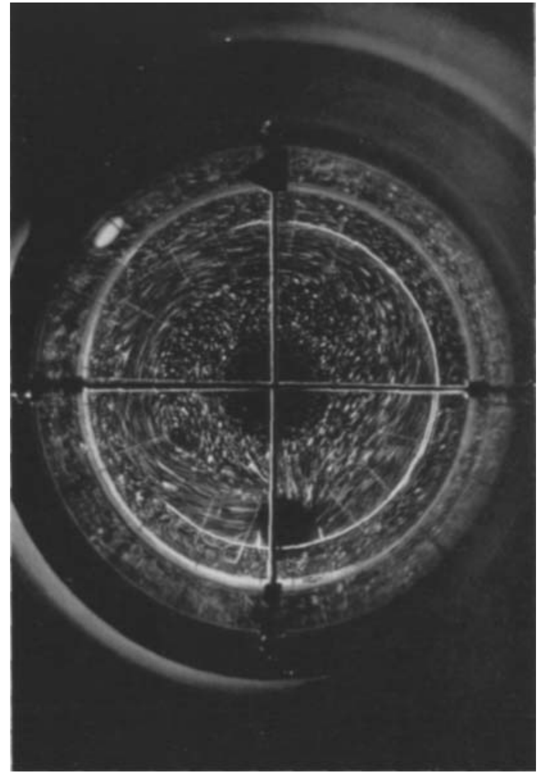

FIGURE 2. The same experiment as that in figure 1 except $\omega > \Omega$. The motion is with respect to the containing vessel so that the obstacle pulls a westerly jet with it.

to a maximum just in the lee. Outside, fluid particles move slowly toward the jet from pole and equator. If we neglect lateral friction, except in the jet, particles bring in the absolute vorticity† of the latitude at which they originate. This means an influx of positive relative vorticity on the poleward side and negative relative vorticity on the equatorial side. This influx can support the strong shear at the edges against the diffusing effect of viscosity.

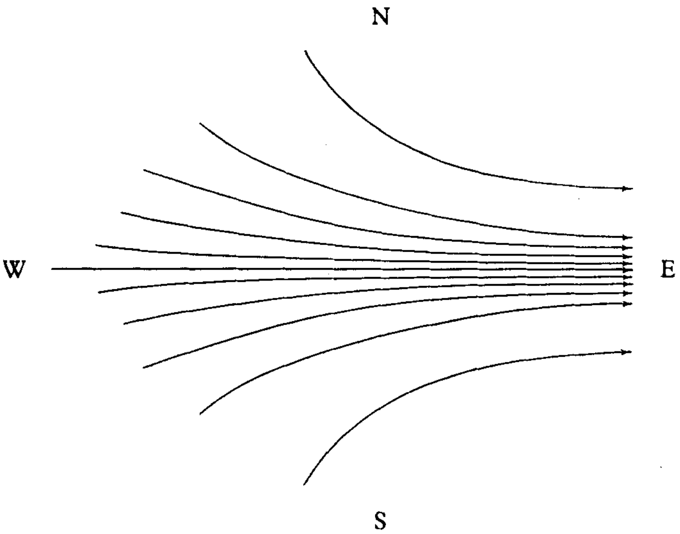

FIGURE 3. Schematic jet.

2. Theory

We approach this problem analytically by assuming a viscous fluid moving two-dimensionally in relative motion parallel to a spherical surface with a basic rotation Ω. Density variations are zero. The radius of the sphere is a, latitude θ, longitude ϕ, the potential of external forces V, density ρ, pressure p, constant viscosity ν, and velocities

$$u^* = a\cos\theta\frac{d\phi}{dt}, \quad v^* = \frac{d\theta}{dt}.$$

The Navier–Stokes equations of motion in the rotating co-ordinate system are

$$\frac{u^*u^*_\phi}{a\cos\theta}+\frac{v^*u^*_\theta}{a}-\frac{u^*v^*\sin\theta}{a\cos\theta}-2\Omega v^*\sin\theta = -\frac{(p/\rho+V)_\phi}{a\cos\theta}$$
$$+\nu\left(\frac{u^*_{\theta\theta}}{a^2}-\frac{\sin\theta}{a^2\cos\theta}u^*_\theta+\frac{u^*_{\phi\phi}}{a^2\cos^2\theta}-\frac{u^*}{a^2\cos^2\theta}-\frac{2\sin\theta}{a^2\cos^2\theta}v^*_\phi\right), \tag{1}$$

$$\frac{u^*v^*_\phi}{a\cos\theta}+\frac{v^*v^*_\theta}{a}+\frac{u^{*2}\sin\theta}{a\cos\theta}+2\Omega u^*\sin\theta = -\frac{(p/\rho+V)_\theta}{a}$$
$$+\nu\left(\frac{v^*_{\theta\theta}}{a^2}-\frac{\sin\theta}{a^2\cos\theta}v^*_\theta+\frac{v^*_{\phi\phi}}{a^2\cos^2\theta}-\frac{v^*}{a^2\cos^2\theta}+\frac{2\sin\theta}{a^2\cos^2\theta}u^*_\phi\right). \tag{2}$$

† We refer to the vorticity component perpendicular to the sphere. This is conserved in frictionless, two-dimensional flow.

The equation of continuity is

$$\frac{u^*_\phi}{a\cos\theta}+\frac{v^*_\theta}{a}-\frac{v^*\sin\theta}{a\cos\theta}=0,$$

so that

$$v^*=\frac{\psi^*_\phi}{a\cos\theta},\quad u^*=-\frac{\psi^*_\theta}{a},$$

where ψ^* is the stream function.

In accordance with the discussion of §1, we assume a jet flow moving west–east at latitude θ_0 with a width of the order δ. It is convenient to adopt two new independent variables x^*, y^*, where x^* is distance along the axis of the jet in the direction of motion and y^* is distance toward the north from latitude θ_0. Let

$$dx^*=a\cos\theta_0\,d\phi,\quad dy^*=a\,d\theta,$$

and expand $\sin\theta$ and $\cos\theta$ in a Taylor series about θ_0,

$$\cos\theta=\cos\theta_0-\sin\theta_0\frac{y^*}{a}-\frac{\cos\theta_0}{2}\frac{y^{*2}}{a^2}+\ldots,$$

$$\sin\theta=\sin\theta_0+\cos\theta_0\frac{y^*}{a}-\frac{\sin\theta_0}{2}\frac{y^{*2}}{a^2}+\ldots.$$

In these expansions y^*/a is very small† in the jet and its vicinity, and from the viewpoint of boundary-layer theory we should at first glance feel justified in writing $\cos\theta=\cos\theta_0$, $\sin\theta=\sin\theta_0$ everywhere in the equations of the problem. We will do this except in the Coriolis force term where we retain the first *two* terms in the expansion of $\sin\theta$. This must be done because the first approximations to the Coriolis terms, $-2\Omega v^*\sin\theta_0$, $2\Omega u^*\sin\theta_0$, can be written

$$-\frac{1}{a\cos\theta_0}\frac{\partial}{\partial\phi}(2\Omega\sin\theta_0\,\psi^*),\quad -\frac{1}{a}\frac{\partial}{\partial\theta}(2\Omega\sin\theta_0\,\psi^*),$$

and are consequently eliminated with the pressure terms when we cross-differentiate to produce the vorticity equation. The vorticity effect of the earth's rotation,‡ discussed in §1, would therefore be lost unless we retained an additional term in the expansion of $\sin\theta$.

Using this approximation and dropping other terms that are small from the viewpoint of boundary-layer theory,§ the equations become

$$u^*u^*_{x^*}+v^*u^*_{y^*}-\beta y^*v^*=-\chi^*_{x^*}+\nu u^*_{y^*y^*}, \tag{3}$$

$$\beta y^*u^*=-\chi^*_{y^*}, \tag{4}$$

$$u^*_{x^*}+v^*_{y^*}=0, \tag{5}$$

† This is certainly true in atmospheric and oceanic jets. The jet of figure 2 is relatively wide and extends around the globe. Our theory which ultimately reduces the globe to a plane, except for the dynamic effect of the variation of angular velocity, may have only a loose connexion with the experimental jet.

‡ Rossby (1939) was the first to use this approximation for the Coriolis parameter $2\Omega\sin\theta$. It has been used frequently in meteorology and oceanography, usually without attempts to justify it.

§ It is an interesting, and perhaps an important fact, that the largest neglected terms in our approximate theory are one order smaller for an equatorial jet than for a jet located at an arbitrary latitude.

where $\beta = 2\Omega\cos\theta_0/a$ and $\chi^* = p/\rho + V - 2\Omega\sin\theta_0\,\psi^*$. The kinematic momentum transfer is

$$J = \int_{-\infty}^{\infty} (\chi^* - \beta y^* \psi^* + u^{*2})\,dy^*. \tag{6}$$

If we require that u^* and its first derivative with respect to y^* approach zero as $|y^*| \to \infty$, J must be constant. We can now eliminate all constants from our problem by defining the non-dimensional quantities

$$u = \frac{u^*}{K^{\frac{2}{5}}\beta^{\frac{1}{5}}}, \quad v = \frac{v^*}{\nu\beta^{\frac{2}{5}}K^{-\frac{1}{5}}}, \quad \chi = \frac{\chi^*}{K^{\frac{4}{5}}\beta^{\frac{2}{5}}},$$

$$\psi = \frac{\psi^*}{K^{\frac{3}{5}}\beta^{-\frac{1}{5}}}, \quad x = \frac{x^*}{\beta^{-\frac{3}{5}}\nu^{-1}K^{\frac{4}{5}}}, \quad y = \frac{y^*}{K^{\frac{1}{5}}\beta^{-\frac{2}{5}}}, \tag{7}$$

where $K = -J$. The resulting equations have the same form as (3)–(6) except that the constants are all missing:

$$uu_x + vu_y - yv = -\chi_x + u_{yy}, \tag{8}$$

$$yu = -\chi_y, \tag{9}$$

$$u_x + v_y = 0, \tag{10}$$

$$-1 = 2\int_0^{\infty} (\chi - y\psi + u^2)\,dy. \tag{11}$$

Thus the problem of the jet can be solved once and for all, and for all conditions, by solving equations (8)–(11). This is not done in this paper although we indicate below a sequence of successive approximations and solve completely the first approximation. At this point we draw some inferences about the Cromwell jet from order of magnitude arguments applied to (8)–(11). Since the boundary conditions imposed on the problem introduce no new constants, we may be confident that all quantities u, v, χ, x, y, entering (8)–(11) are of order of magnitude one. Assuming this and applying equation (7) we evaluate certain quantities of interest in the special case of the equatorial undercurrent. If we take $1{\cdot}5 \times 10^2\,\text{cm}\,\text{sec}^{-1}$ as an observed speed at some point on the axis of the jet and $\beta = 2\Omega/a \sim 2{\cdot}3 \times 10^{-13}\,\text{cm}^{-1}\,\text{sec}^{-1}$, the first relation in (7) yields

$$K \sim 5{\cdot}7 \times 10^{11}\,\text{cm}^3\,\text{sec}^{-2}.$$

Then

$$\delta \sim 2{\cdot}6 \times 10^7\,\text{cm}$$

is an estimate of the width of the current. This is of the same order as the observed width and the numerical value should be compared with a calculation below using the linearized equations. It is remarkable that the above estimate does not require any knowledge of the assumed coefficient of friction. It can be verified from the linearized theory that the phenomenon as a whole is very insensitive to the magnitude of the friction.† A further check on the order of magnitude of quantities in the undercurrent is in the computation of the horizontal length scale from (7). We get

$$L \sim \frac{10^{17}}{\nu}\,\text{cm}^3\,\text{sec}^{-1}.$$

† One might also guess that the phenomenon is insensitive to the exact form of the law of friction. This would be important if true because the eddy viscosity concept, especially with a constant coefficient, is admittedly crude.

If we use a figure of $7 \times 10^7\,\mathrm{cm^2\,sec^{-1}}$ for ν, suggested by Montgomery & Palmén (1940), we find $L \sim 1{\cdot}4 \times 10^9$ cm or perhaps 8000 miles. This seems at first glance a little large, by a factor of two or three, but not if we append the observation that measurements show little or no variation of the current over a distance of 3000 miles or so.

Equations (8)–(11) may be put in a better form by using as dependent variables ψ and Λ, the latter being defined by

$$\begin{aligned} \Lambda_y &= -2(\chi - y\psi + u^2), \\ \Lambda(x, 0) &= 0. \end{aligned} \tag{12}$$

Then the problem is determined by

$$\Lambda_x - 2\psi_{yy} + 2\psi_y \psi_x = 0, \tag{13}$$

$$2\psi - \Lambda_{yy} - 4\psi_y \psi_{yy} = 0, \tag{14}$$

$$\Lambda(x, \infty) = 1, \tag{15}$$

together with the conditions of symmetry and finiteness.

We may approach the solution of (13)–(15) by noticing that far enough upstream the velocities in the jet are arbitrarily small and only linear terms are important. The linear problem has solutions of the form

$$\Lambda_0 = h_0(z),$$

$$\psi_0 = (-x)^{-\frac{2}{4}} l_0(z),$$

where

$$z = (-x)^{-\frac{1}{4}} y.$$

We are led then to a scheme of successive approximations

$$\Lambda = \sum_{n=0}^{\infty} (-x)^{-\frac{5}{4}n} h_n(z), \tag{16}$$

$$\psi = \sum_{n=0}^{\infty} (-x)^{-\frac{1}{4}(2+5n)} l_n(z). \tag{17}$$

Substituting (16) and (17) into (13) and (14), we obtain sets of equations in the single independent variable z. The first set is

$$\begin{aligned} 8l_0'' - h_0' z &= 0, \\ h_0'' - 2l_0 &= 0, \end{aligned} \tag{18}$$

where the prime denotes differentiation with respect to z.

3. First approximation

The system (18) may be solved very simply by expanding in a Taylor series about $z = 0$. After satisfying the symmetry and finiteness conditions at $z = 0$ two constants remain. One may be determined in terms of the other by imposing the finiteness condition at $z = \infty$. The single remaining constant can be found by satisfying $h_0(\infty) = 1$. In the numerical computation, one obtains sufficient accuracy by assuming $l_0' = 0$ at $z = 10$ or so, since imposition of the condition at both $z = 10$ and $z = 8$ does not give significantly different results except near the extreme value of z. The solution is shown in figure 4.

The u-component reveals counter currents on both sides of the main jet with maximum speed about one-half that of the primary jet. The motion dies off rapidly further north and south. The width of the primary jet is given approximately by

$$\delta \sim 4(-x^*)^{\frac{1}{4}} \nu^{\frac{1}{4}} \beta^{-\frac{1}{4}}. \tag{19}$$

It narrows very gradually downstream. The width increases with the viscosity and decreases with higher rotation, but very insensitively. In the ocean at $x^* = 5 \times 10^8$ cm, the width is 8×10^7 cm, more than twice that of the undercurrent.

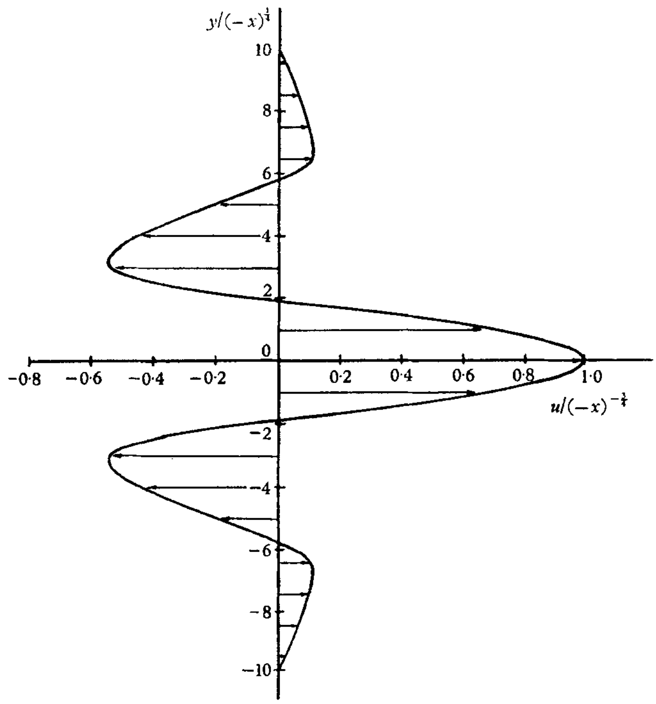

FIGURE 4. Velocity profile in the linear portion of jet. Since we have not imposed the condition that $h_0(\infty) = 1$, the abscissa scale may be multiplied by an arbitrary constant.

4. Non-linearity

The solution of (8)–(11) in regions where the non-linear terms are important has not yet been found. In the case of the equatorial undercurrent the non-linear terms will be unimportant if, for example, $v^* u^*_{y^*} / \beta y^* v^* \ll 1$, or

$$B = \frac{u^*}{\beta \delta^2} \ll 1.$$

Using $u^* \sim 1{\cdot}5 \times 10^2\,\mathrm{cm\,sec^{-1}}$, $\beta \sim 2{\cdot}3 \times 10^{-13}\,\mathrm{cm^{-1}\,sec^{-1}}$, $\delta \sim 2{\cdot}6 \times 10^7\,\mathrm{cm}$, we see that $B \sim 1$ and the condition is not met. The fact that the non-linear terms are not overriding (as they are, for example, in atmospheric jet phenomena) indicates that the ocean current may be qualitatively similar to that in the linear case. The effect of non-linearity can be judged by considering the physical picture of the maintenance of the jets by a balance of advection and diffusion of vorticity. We see that in the linear solution the jet system narrows as β increases. The non-linear terms serve to bring in additional perturbed vorticity and their inclusion should have effects similar to those associated with an increase of β, namely, a narrowing of the jet. The modification is in the desired direction because the linear current turned out to be too wide to fit the observations of the undercurrent.

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